Preliminaries

This chapter contains a variety of definitions of terms that are used throughout the rest of the book, but are not part of linear algebra and/or do not fit naturally into another chapter. Since these definitions have little connection with each other, a different organization is followed; the definitions are (loosely) alphabetized and each definition is followed by an example.

Algebra

An **(associative) algebra** is a vector space A over a field F together with a **multiplication** $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{xy}$ from $A \times A$ to A satisfying two *distributive* properties and *associativity*, i.e., for all $a, b \in F$ and all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in A$:

$$(a\mathbf{x} + b\mathbf{y})\mathbf{z} = a(\mathbf{xz}) + b(\mathbf{yz}), \qquad \mathbf{x}(a\mathbf{y} + b\mathbf{z}) = a(\mathbf{xy}) + b(\mathbf{xz}) \qquad (\mathbf{xy})\mathbf{z} = \mathbf{x}(\mathbf{yz}).$$

Except in Chapter 69 and Chapter 70 the term *algebra* means associative algebra. In these two chapters, associativity is not assumed.

Examples:

The vector space of $n \times n$ matrices over a field F with matrix multiplication is an (associative) algebra.

Boundary

The **boundary** ∂S of a subset S of the real numbers or the complex numbers is the intersection of the closure of S and the closure of the complement of S.

Examples:

The boundary of $S = \{x \in \mathbb{C} : |z| \leq 1\}$ is $\partial S = \{x \in \mathbb{C} : |z| = 1\}$.

Complement

The **complement** of the set X in universe S, denoted $S \setminus X$, is all elements of S that are not in X. When the universe is clear (frequently the universe is $\{1, \ldots, n\}$) then this can be denoted X^c.

Examples:

For $S = \{1, 2, 3, 4, 5\}$ and $X = \{1, 3\}$, $S \setminus X = \{2, 4, 5\}$.

Complex Numbers

Let $a, b \in \mathbb{R}$. The symbol i denotes $\sqrt{-1}$.

The **complex conjugate** of a complex number $c = a + bi$ is $\overline{c} = a - bi$.
The **imaginary part** of $a + bi$ is $\text{im}(a + bi) = b$ and the **real part** is $\text{re}(a + bi) = a$.
The **absolute value** of $c = a + bi$ is $|c| = \sqrt{a^2 + b^2}$.

The **argument** of the nonzero complex number $re^{i\theta}$ is θ (with $r, \theta \in \mathbb{R}$ and $0 < r$ and $0 \leq \theta < 2\pi$).
The **open right half plane** $\mathbb{C}^+$ is $\{z \in \mathbb{C} : \text{re}(z) > 0\}$.
The **closed right half plane** $\mathbb{C}_0^+$ is $\{z \in \mathbb{C} : \text{re}(z) \geq 0\}$.
The **open left half plane** $\mathbb{C}^-$ is $\{z \in \mathbb{C} : \text{re}(z) < 0\}$.
The **closed left half plane** $\mathbb{C}^-$ is $\{z \in \mathbb{C} : \text{re}(z) \leq 0\}$.

Facts:

1. $|c| = c\overline{c}$
2. $|re^{i\theta}| = r$
3. $re^{i\theta} = r\cos\theta + r\sin\theta i$
4. $\overline{re^{i\theta}} = re^{-i\theta}$

Examples:

$\overline{2+3i} = 2 - 3i$, $\overline{1.4} = 1.4$, $1 + i = \sqrt{2}e^{i\pi/4}$.

Conjugate Partition

Let $\upsilon = (u_1, u_2, \ldots, u_n)$ be a sequence of integers such that $u_1 \geq u_2 \geq \cdots \geq u_n \geq 0$. The **conjugate partition** of υ is $\upsilon^* = (u_1^*, \ldots, u_t^*)$, where u_i^* is the number of js such that $u_j \geq i$. t is sometimes taken to be u_1, but is sometimes greater (obtained by extending with 0s).

Facts: If t is chosen to be the minimum, and $u_n > 0$, $\upsilon^{**} = \upsilon$.

Examples:

$(4, 3, 2, 2, 1)^* = (5, 4, 2, 1)$.

Convexity

Let V be a real or complex vector space.

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \in V$. A vector of the form $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$ with all the coefficients a_i nonnegative and $\sum a_i = 1$ is a **convex combination** of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

A set $S \subseteq V$ is **convex** if any convex combination of vectors in S is in S.

The **convex hull** of S is the set of all convex combinations of S and is denoted by $\text{Con}(S)$.

An **extreme point** of a closed convex set S is a point $\mathbf{v} \in S$ that is not a nontrivial convex combination of other points in S, i.e., $a\mathbf{x} + (1-a)\mathbf{y} = \mathbf{v}$ and $0 \leq a \leq 1$ implies $\mathbf{x} = \mathbf{y} = \mathbf{v}$.

A **convex polytope** is the convex hull of a finite set of vectors in $\mathbb{R}^n$.

Let $S \subseteq V$ be convex. A function $f : S \to \mathbb{R}$ is **convex** if for all $a \in \mathbb{R}, 0 < a < 1, \mathbf{x}, \mathbf{y} \in S,\ f(a\mathbf{x} + (1-a)\mathbf{y}) \leq af(\mathbf{x}) + (1-a)f(\mathbf{y})$.

Facts:

1. A set $S \subseteq V$ is convex if and only if $\text{Con}(S) = S$.
2. The extreme points of $\text{Con}(S)$ are contained in S.
3. [HJ85] *Krein-Milman Theorem*: A compact convex set is the convex hull of its extreme points.

Examples:

1. $[1.9, 0.8]^T$ is a convex combination of $[1, -1]^T$ and $[2, 1]^T$, since $[1.9, 0.8]^T = 0.1[1, -1]^T + 0.9[2, 1]^T$.
2. The set K of all $\mathbf{v} \in \mathbb{R}^3$ such that $v_i \geq 0, i = 1, 2, 3$ is a convex set. Its only extreme point is the zero vector.

Elementary Symmetric Function

The k**th elementary symmetric function of** $\alpha_i, i = 1, \ldots, n$ is

$$S_k(\alpha_1, \ldots, \alpha_n) = \sum_{1<i_1<i_2<\cdots<i_k<n} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k}.$$

Examples:

$S_2(\alpha_1, \alpha_2, \alpha_3) = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$,
$S_1(\alpha_1, \ldots, \alpha_n) = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $S_n(\alpha_1, \ldots, \alpha_n) = \alpha_1\alpha_2\ldots\alpha_n$.

Equivalence Relation

A binary relation $\equiv$ in a nonempty set S is an **equivalence relation** if it satisfies the following conditions:

1. (Reflexive) For all $a \in S$, $a \equiv a$.
2. (Symmetric) For all $a, b \in S$, $a \equiv b$ implies $b \equiv a$.
3. (Transitive) For all $a, b, c \in S$, $a \equiv b$ and $a \equiv b$ imply $a \equiv c$.

Examples:

Congruence mod n is an equivalence relation on the integers.

Field

A **field** is a set F with at least two elements together with a function $F \times F \to F$ called addition, denoted $(a, b) \to a + b$, and a function $F \times F \to F$ called multiplication, denoted $(a, b) \to ab$, which satisfy the following axioms:

1. (Commutativity) For each $a, b \in F$, $a + b = b + a$ and $ab = ba$.
2. (Associativity) For each $a, b, c \in F$, $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
3. (Identities) There exist two elements 0 and 1 in F such that $0 + a = a$ and $1a = a$ for each $a \in F$.
4. (Inverses) For each $a \in F$, there exists an element $-a \in F$ such that $(-a) + a = 0$. For each nonzero $a \in F$, there exists an element $a^{-1} \in F$ such that $a^{-1}a = 1$.
5. (Distributivity) For each $a, b, c \in F, a(b + c) = ab + ac$.

Examples:

The real numbers, $\mathbb{R}$, the complex numbers, $\mathbb{C}$, and the rational numbers, $\mathbb{Q}$, are all fields. The set of integers, $\mathbb{Z}$, is not a field.

Greatest Integer Function

The **greatest integer** or **floor** function $\lfloor x \rfloor$ (defined on the real numbers) is the greatest integer less than or equal to x.

Examples:

$\lfloor 1.5 \rfloor = 1$, $\lfloor 1 \rfloor = 1$, $\lfloor -1.5 \rfloor = -2$.

Group

(See also Chapter 67 and Chapter 68.)

A **group** is a nonempty set G with a function $G \times G \to G$ denoted $(a, b) \to ab$, which satisfies the following axioms:

1. (Associativity) For each $a, b, c \in G$, $(ab)c = a(bc)$.
2. (Identity) There exists an element $e \in G$ such that $ea = a = ae$ for each $a \in G$.
3. (Inverses) For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a^{-1}a = e = aa^{-1}$.

A group is **abelian** if $ab = ba$ for all $a, b \in G$.

Examples:

1. Any vector space is an abelian group under $+$.
2. The set of invertible $n \times n$ real matrices is a group under matrix multiplication.
3. The set of all permutations of a set is a group under composition.

Interlaces

Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_{n-1}$, two sequences of real numbers arranged in decreasing order. Then the sequence $\{b_i\}$ **interlaces** the sequence $\{a_i\}$ if $a_n \leq b_{n-1} \leq a_{n-1} \cdots \leq b_1 \leq a_1$. Further, if all of the above inequalities can be taken to be strict, the sequence $\{b_i\}$ **strictly interlaces** the sequence $\{a_i\}$. Analogous definitions are given when the numbers are in increasing order.

Examples:

$7 \geq 2.2 \geq -1$ strictly interlaces $11 \geq \pi \geq 0 \geq -2.6$.

Majorization

Let $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_n)$ be sequences of real numbers.

$\alpha^{\downarrow} = (a_1^{\downarrow}, a_2^{\downarrow}, \ldots, a_n^{\downarrow})$ is the permutation of α with entries in nonincreasing order, i.e., $a_1^{\downarrow} \geq a_2^{\downarrow} \geq \ldots \geq a_n^{\downarrow}$.

$\alpha^{\uparrow} = (a_1^{\uparrow}, a_2^{\uparrow}, \ldots, a_n^{\uparrow})$ is the permutation of α with entries in nondecreasing order, i.e., $a_1^{\uparrow} \leq a_2^{\uparrow} \leq \ldots \leq a_n^{\uparrow}$.

α **weakly majorizes** β, written $\alpha \succeq_w \beta$ or $\beta \preceq_w \alpha$, if:

$$\sum_{i=1}^{k} a_i^{\downarrow} \geq \sum_{i=1}^{k} b_i^{\downarrow} \quad \text{for all } k = 1, \ldots n.$$

α **majorizes** β, written $\alpha \succeq \beta$ or $\beta \preceq \alpha$, if $\alpha \succeq_w \beta$ and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$.

Examples:

1. If $\alpha = (2, 2, -1.3, 8, 7.7)$, then $\alpha^{\downarrow} = (8, 7.7, 2, 2, -1.3)$ and $\alpha^{\uparrow} = (-1.3, 2, 2, 7.7, 8)$.
2. $(5,3,1.5,1.5,1) \succeq (4,3,2,2,1)$ and $(6,5,0) \succeq_w (4,3,2)$.

Metric

A **metric** on a set S is a real-valued function $f : S \times S \to \mathbb{R}$ satisfying the following conditions:

1. For all $x, y \in S$, $f(x, y) \geq 0$.
2. For all $x \in S$, $f(x, x) = 0$.
3. For all $x, y \in S$, $f(x, y) = 0$ implies $x = y$.
4. For all $x, y \in S$, $f(x, y) = f(y, x)$.
5. For all $x, y, z \in S$, $f(x, y) + f(y, z) \geq f(x, z)$.

A metric is intended as a measure of distance between elements of the set.

Examples:

If $\| \cdot \|$ is a norm on a vector space, then $f(x, y) = \|\mathbf{x} - \mathbf{y}\|$ is a metric.

Multiset

A **multiset** is an unordered list of elements that allows repetition.

Examples:

Any set is a multiset, but $\{1, 1, 3, -2, -2, -2\}$ is a multiset that is not a set.

O and o

Let, f, g be real valued functions of $\mathbb{N}$ or $\mathbb{R}$, i.e., $f, g : \mathbb{N} \to \mathbb{R}$ or $f, g : \mathbb{R} \to \mathbb{R}$.

f is $O(g)$ (**big-oh** of g) if there exist constants C, k such that $|f(x)| \leq C|g(x)|$ for all $x \geq k$.

f is $o(g)$ (**little-oh** of g) if $\lim_{x\to\infty} \left| \frac{f(n)}{g(n)} \right| = 0$.

Examples:

$x^2 + x$ is $O(x^2)$ and $\ln x$ is $o(x)$.

Path-connected

A subset S of the complex numbers is **path-connected** if for any $x, y \in S$ there exists a continuous function $p : [0, 1] \to S$ with $p(0) = x$ and $p(1) = y$.

Examples:

$S = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$ and the line $\{a + bi : a = 2b + 3\}$ are path-connected.

Permutations

A **permutation** is a one-to-one onto function from a set to itself.

The set of permutations of $\{1, \ldots, n\}$ is denoted S_n. The identity permutation is denoted ε_n. In this book, permutations are generally assumed to be elements of S_n for some n.

A **cycle** or k-**cycle** is a permutation τ such that there is a subset $\{a_1, \ldots, a_k\}$ of $\{1, \ldots, n\}$ satisfying $\tau(a_i) = a_{i+1}$ and $\tau(a_k) = a_1$; this is denoted $\tau = (a_1, a_2, \ldots, a_k)$. The **length** of this cycle is k.

A **transposition** is a 2-cycle.

A permutation is **even** (respectively, **odd**) if it can be written as the product of an even (odd) number of transpositions.

The **sign** of a permutation τ, denoted sgn τ, is +1 if τ is even and -1 if τ is odd.

Note: Permutations are functions and act from the left (see Examples).

Facts:

1. Every permutation can be expressed as a product of disjoint cycles. This expression is unique up to the order of the cycles in the decomposition and cyclic permutation within a cycle.
2. Every permutation can be written as a product of transpositions. If some such expression includes an even number of transpositions, then every such expression includes an even number of transpositions.
3. S_n with the operation of composition is a group.

Examples:

1. If $\tau = (1523) \in S_6$, then $\tau(1) = 5, \tau(2) = 3, \tau(3) = 1, \tau(4) = 4, \tau(5) = 2, \tau(6) = 6$.
2. (123)(12)=(13).
3. sgn(1234) = −1, because (1234) = (14)(13)(12).

Ring

(See also Section 23.1)
A **ring** is a set R together with a function $R \times R \to R$ called addition, denoted $(a, b) \to a+b$, and a function $R \times R \to R$ called multiplication, denoted $(a, b) \to ab$, which satisfy the following axioms:

1. (Commutativity of +) For each $a, b \in R$, $a + b = b + a$.
2. (Associativity) For each $a, b, c \in R$, $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
3. (+ identity) There exists an element 0 in R such that $0 + a = a$.
4. (+ inverse) For each $a \in R$, there exists an element $-a \in R$ such that $(-a) + a = 0$.
5. (Distributivity) For each $a, b, c \in R, a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

A **zero divisor** in a ring R is a nonzero element $a \in R$ such that there exists a nonzero $b \in R$ with $ab = 0$ or $ba = 0$.

Examples:

- The set of integers, $\mathbb{Z}$, is a ring.
- Any field is a ring.
- Let F be a field. Then $F^{n\times n}$, with matrix addition and matrix multiplication as the operations, is a ring. $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are zero divisors since $E_{11}E_{22} = 0_2$.

Sign

(For sign of a permutation, see *permutation*.)

The **sign** of a complex number is defined by:

$$\operatorname{sign}(z) = \begin{cases} z/|z|, & \text{if } z \neq 0; \\ 1, & \text{if } z = 0. \end{cases}$$

If z is a real number, this sign function yields 1 or −1.

This sign function is used in numerical linear algebra.

The **sign** of a real number (as used in sign patterns) is defined by:

$$\operatorname{sgn}(a) = \begin{cases} +, & \text{if } a > 0; \\ 0, & \text{if } a = 0; \\ -, & \text{if } a < 0. \end{cases}$$

This sign function is used in combinatorial linear algebra, and the product of a sign and a real number is interpreted in the obvious way as a real number.

Warning: The two sign functions disagree on the sign of 0.

Examples:

$\operatorname{sgn}(-1.3) = -$, $\operatorname{sign}(-1.3) = -1$, $\operatorname{sgn}(0) = 0$, $\operatorname{sign}(0) = 1$,
$\operatorname{sign}(1 + i) = \dfrac{(1 + i)}{\sqrt{2}}$.

References

[HJ85] [HJ85] R. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, 1985.